HELPFUL HANDOUTS

Geometric Mean

Geometric Mean The **geometric mean** between two numbers is the positive square root of their product. For two positive numbers a and b, the geometric mean of a and b is the positive number x in the proportion $\frac{a}{x} = \frac{x}{b}$. Cross multiplying gives $x^2 = ab$, so $x = \sqrt{ab}$.

Example **Find the geometric mean between each pair of numbers.**

a. 12 and 3

Let x represent the geometric mean.

$\frac{12}{x} = \frac{x}{3}$	Definition of geometric mean
$x^2 = 36$	Cross multiply.
$x = \sqrt{36}$ or 6	Take the square root of each side.

b. 8 and 4

Let x represent the geometric mean.

$\frac{8}{x} = \frac{x}{4}$

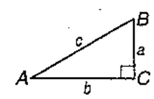
$x^2 = 32$

$x = \sqrt{32}$

≈ 5.7

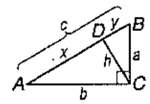
The Pythagorean Theorem and Its Converse

The Pythagorean Theorem In a right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.



$\triangle ABC$ is a right triangle, so $a^2 + b^2 = c^2$.

Example 1 **Prove the Pythagorean Theorem.**



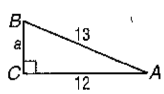
With altitude $\overline{CD}$, each leg a and b is a geometric mean between hypotenuse c and the segment of the hypotenuse adjacent to that leg.

$\frac{c}{a} = \frac{a}{y}$ and $\frac{c}{b} = \frac{b}{x}$, so $a^2 = cy$ and $b^2 = cx$.

Add the two equations and substitute $c = y + x$ to get $a^2 + b^2 = cy + cx = c(y + x) = c^2$.

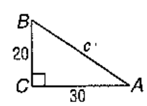
Example 2

a. Find a.



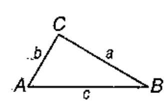
$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 12^2 = 13^2$	$b = 12$, $c = 13$
$a^2 + 144 = 169$	Simplify.
$a^2 = 25$	Subtract.
$a = 5$	Take the positive square root of each side.

b. Find c.



$a^2 + b^2 = c^2$	Pythagorean Theorem
$20^2 + 30^2 = c^2$	$a = 20$, $b = 30$
$400 + 900 = c^2$	Simplify.
$1300 = c^2$	Add.
$\sqrt{1300} = c$	Take the positive square root of each side.
$36.1 \approx c$	Use a calculator.

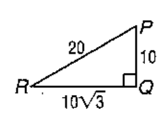
Converse of the Pythagorean Theorem If the sum of the squares of the measures of the two shorter sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.



If the three whole numbers a, b, and c satisfy the equation $a^2 + b^2 = c^2$, then the numbers a, b, and c form a **Pythagorean triple**.

If $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.

Example **Determine whether $\triangle PQR$ is a right triangle.**



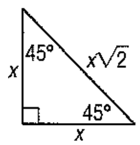
$a^2 + b^2 \stackrel{?}{=} c^2$	Pythagorean Theorem
$10^2 + (10\sqrt{3})^2 \stackrel{?}{=} 20^2$	$a = 10$, $b = 10\sqrt{3}$, $c = 20$
$100 + 300 \stackrel{?}{=} 400$	Simplify.
$400 = 400 \checkmark$	Add.

The sum of the squares of the two shorter sides equals the square of the longest side, so the triangle is a right triangle.

Special Right Triangles

Properties of 45°-45°-90° Triangles The sides of a 45°-45°-90° right triangle have a special relationship.

Example 1 **If the leg of a 45°-45°-90° right triangle is x units, show that the hypotenuse is $x\sqrt{2}$ units.**



Using the Pythagorean Theorem with $a = b = x$, then

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= x^2 + x^2 \\ &= 2x^2 \\ c &= \sqrt{2x^2} \\ &= x\sqrt{2} \end{aligned}$$

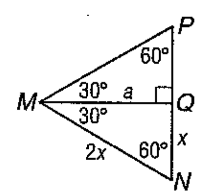
Example 2 **In a 45°-45°-90° right triangle the hypotenuse is $\sqrt{2}$ times the leg. If the hypotenuse is 6 units, find the length of each leg.**

The hypotenuse is $\sqrt{2}$ times the leg, so divide the length of the hypotenuse by $\sqrt{2}$.

$$\begin{aligned} a &= \frac{6}{\sqrt{2}} \\ &= \frac{6\sqrt{2}}{\sqrt{2}\sqrt{2}} \\ &= \frac{6\sqrt{2}}{2} \\ &= 3\sqrt{2} \text{ units} \end{aligned}$$

Properties of 30°-60°-90° Triangles The sides of a 30°-60°-90° right triangle also have a special relationship.

Example 1 **In a 30°-60°-90° right triangle, show that the hypotenuse is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.**



$\triangle MNQ$ is a 30°-60°-90° right triangle, and the length of the hypotenuse $\overline{MN}$ is two times the length of the shorter side $\overline{NQ}$. Using the Pythagorean Theorem,

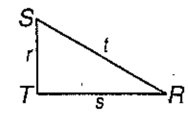
$$\begin{aligned} a^2 &= (2x)^2 - x^2 \\ &= 4x^2 - x^2 \\ &= 3x^2 \\ a &= \sqrt{3x^2} \\ &= x\sqrt{3} \end{aligned}$$

Example 2 **In a 30°-60°-90° right triangle, the hypotenuse is 5 centimeters. Find the lengths of the other two sides of the triangle.**

If the hypotenuse of a 30°-60°-90° right triangle is 5 centimeters, then the length of the shorter leg is half of 5 or 2.5 centimeters. The length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, or $(2.5)(\sqrt{3})$ centimeters.

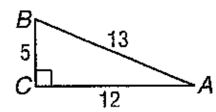
Trigonometry

Trigonometric Ratios The ratio of the lengths of two sides of a right triangle is called a **trigonometric ratio**. The three most common ratios are **sine**, **cosine**, and **tangent**, which are abbreviated *sin*, *cos*, and *tan*, respectively.



$$\sin R = \frac{\text{leg opposite } \angle R}{\text{hypotenuse}} = \frac{r}{t} \qquad \cos R = \frac{\text{leg adjacent to } \angle R}{\text{hypotenuse}} = \frac{s}{t} \qquad \tan R = \frac{\text{leg opposite } \angle R}{\text{leg adjacent to } \angle R} = \frac{r}{s}$$

Example **Find sin *A*, cos *A*, and tan *A*. Express each ratio as a decimal to the nearest thousandth.**



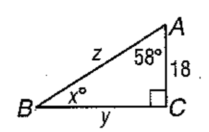
$$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{5}{13} \approx 0.385$$

$$\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{12}{13} \approx 0.923$$

$$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{BC}{AC} = \frac{5}{12} \approx 0.417$$

Use Trigonometric Ratios In a right triangle, if you know the measures of two sides or if you know the measures of one side and an acute angle, then you can use trigonometric ratios to find the measures of the missing sides or angles of the triangle.

Example **Find x, y, and z. Round each measure to the nearest whole number.**



a. Find x.

$$x + 58 = 90$$
$$x = 32$$

b. Find y.

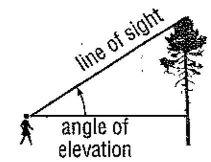
$$\tan A = \frac{y}{18}$$
$$\tan 58° = \frac{y}{18}$$
$$y = 18 \tan 58°$$
$$y \approx 29$$

c. Find z.

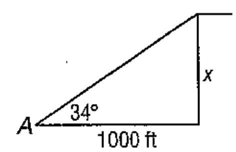
$$\cos A = \frac{18}{z}$$
$$\cos 58° = \frac{18}{z}$$
$$z \cos 58° = 18$$
$$z = \frac{18}{\cos 58°}$$
$$z \approx 34$$

Angles of Elevation and Depression

Angles of Elevation Many real-world problems that involve looking up to an object can be described in terms of an **angle of elevation**, which is the angle between an observer's line of sight and a horizontal line.



Example **The angle of elevation from point A to the top of a cliff is 34°. If point A is 1000 feet from the base of the cliff, how high is the cliff?**



Let x = the height of the cliff.

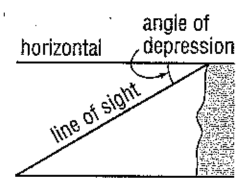
$$\tan 34° = \frac{x}{1000} \qquad \tan = \frac{\text{opposite}}{\text{adjacent}}$$

$$1000(\tan 34°) = x \qquad \text{Multiply each side by 1000.}$$

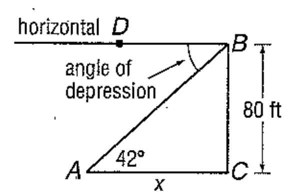
$$674.5 = x \qquad \text{Use a calculator.}$$

The height of the cliff is about 674.5 feet.

Angles of Depression When an observer is looking down, the **angle of depression** is the angle between the observer's line of sight and a horizontal line.



Example **The angle of depression from the top of an 80-foot building to point A on the ground is 42°. How far is the foot of the building from point A?**



Let x = the distance from point A to the foot of the building. Since the horizontal line is parallel to the ground, the angle of depression $\angle DBA$ is congruent to $\angle BAC$.

$$\tan 42° = \frac{80}{x} \qquad \tan = \frac{\text{opposite}}{\text{adjacent}}$$

$$x(\tan 42°) = 80 \qquad \text{Multiply each side by } x.$$

$$x = \frac{80}{\tan 42°} \qquad \text{Divide each side by } \tan 42°.$$

$$x \approx 88.8 \qquad \text{Use a calculator.}$$

Point A is about 89 feet from the base of the building.